* When we multiply any matrix by identity matrix, the matrix does not change

eg. $\begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix}$

The transpose of a matrix

$A = \begin{pmatrix} 2 & 1 & 3 & 5 \\ 0 & -3 & -2 & 4 \end{pmatrix}$ $\quad A^T_{4\times 2} = \begin{pmatrix} 2 & 0 \\ 1 & -3 \\ 3 & -2 \\ 5 & 4 \end{pmatrix}$

2×4

Properties

1- $(A^T)^T = A$

2- $(A+B)^T = A^T + B^T$

3- $(AB)^T = B^T A^T$

4- if $A = A^T \Rightarrow$ symmetric matrix

if $A = -A^T \Rightarrow$ Skew-symmetric matrix

Inverse of a matrix

$AX = B \quad , \quad X = A^{-1}B$

To find inverse of a matrix:

① Matrix must be square matrix, eg. 2×2, 3×3, 4×4 ...

② Not all square matrices have inverse

Find the inverse of the matrix

$A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{pmatrix}$

Gaus - Gordon elimination method

Step 1 : $\left(\underset{3\times3}{A} \,\vdots\, I_3\right)$ Adjoint matrix

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right)$$

Step 2: $\left(I_3 \,\vdots\, A^{-1}\right)$ use row operations to transform matrix A into identity matrix I, the right side of the matrix will be the inverse

$$\xrightarrow[R_3+6R_1]{R_2-R_1} \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -4 & 3 & 6 & 0 & 1 \end{array}\right) \xrightarrow[R_3+4R_2]{R_1+R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 4 & 1 \end{array}\right)$$

$$\xrightarrow{(-1)R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right) \xrightarrow[R_2+R_3]{R_1+R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & -1 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{pmatrix}$$

To check : $A A^{-1} = I$

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -4 & 3 \end{pmatrix} \begin{pmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Find the inverse of Matrix A (if possible)

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ -2 & 3 & -2 \end{pmatrix} \qquad \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 3 & -1 & 2 & 0 & 1 & 0 \\ -2 & 3 & -2 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3+2R_1]{R_2-3R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 3 & -7 & 2 & -3 & 1 & 0 \\ 0 & 7 & -2 & 2 & 0 & 1 \end{array}\right)$$

$$\Longrightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -7 & 2 & -3 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right)$$

if you get a row of zeros then there is no inverse.

Special case: Inverse of a 2x2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad A^{-1} = \frac{1}{(ad-bc)} \begin{pmatrix} d & -b \\ c & a \end{pmatrix}$$

Ex $$A = \begin{pmatrix} 3 & -1 \\ -2 & 2 \end{pmatrix} \qquad A^{-1} = \frac{1}{(3\times2)-(-1)\times(-2)} \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{pmatrix}$$

$$B = \begin{pmatrix} 3 & -1 \\ -6 & 2 \end{pmatrix} \qquad B^{-1} = \frac{1}{\cancel{3\times2-6}\ 0} \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix} \Rightarrow \text{no inverse}$$

Properties of inverse

1) $(A^{-1})^{-1} = A$
2) $(A^k)^{-1} = (A^{-1})^k \qquad (A^4)^{-1} = (A^{-1})^4$
3) $(A^T)^{-1} = (A^{-1})^T$
4) $(AB)^{-1} = B^{-1}A^{-1}$

$$2x + 3y + z = -1$$
$$3x + 3y + z = 1$$
$$2x + 4y + z = -2$$

$$\Longrightarrow \begin{pmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$$

$AX = B$

A: Coefficient matrix

X: unknown matrix

B: Constant term matrix

$AX = B$

$\underbrace{A^{-1}A}_{1} X = A^{-1} B$ $\qquad \boxed{X = A^{-1} B}$

Theorem: If A is an invertible matrix (has inverse) then the system of linear equations $AX = B$ has a unique solution given by $X = A^{-1}B$

$$\left(\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right) \Longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 6 & -2 & -3 \end{array}\right)$$

$$X = A^{-1}B = \cancel{-1 \ \ 1 \ \ 0} \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$$

$x = +2 \quad y = -1 \quad z = -2$

Elementary matrices

An $n \times n$ matrix is called an elementary matrix if it can be obtained from the identity matrix I_n by a single operation

eg.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xleftarrow{R_2 \times 8} I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Determine the elementary matrices

① $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ Elementary

② $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ not elementary

③ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ not elementary

④ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ Elementary

⑤ $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ Elementary

⑥ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ not elementary

Find.

① $$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 & 1 \\ 1 & -3 & 6 \\ 3 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -3 & 6 \\ 0 & 2 & 1 \\ 3 & 2 & -1 \end{pmatrix}$$

② $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -4 & 1 \\ 0 & 2 & 6 & -4 \\ 0 & 1 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -4 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 1 & 3 & 1 \end{pmatrix}$$

③ $$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ -2 & -2 & 3 \\ 0 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 0 & 4 & 5 \end{pmatrix}$$

When multiplying matrix A with elementary matrix E, the single operation that transformed identity matrix into E is used on A

Note: The elementary matrix should be on the left side.

The inverse of a matrix

Theorem: If E is an elementary matrix, then E^{-1} exists and is an elementary matrix

Steps to find inverse of elementary matrix

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{pmatrix} \qquad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

To get inverse of E reverse the operation used on I to transform it to E and use it on I

$$I \xrightarrow{9R_3} E$$

$$E \Longrightarrow \quad I \xrightarrow{\frac{1}{9}R_3} E^{-1}$$

$$E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{9} \end{pmatrix}$$

$$E = \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad R_1 + 5R_3$$

$$E^{-1} \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad R_1 - 5R_3$$

	Operation	Inverse operation
1.	Interchange rows p, q	Interchange rows p, q
2.	multiply row p by $c \neq 0$	multiply row p by $\frac{1}{c}$
3.	Add k times row p to row q	Subtract k times row p from row q